Nonlinear Themoacoustic In/Stability of a Rijke Tube with a Distributed Heat Source

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Abstract

In this paper, thermoacoustic dynamic behaviour patterns of a Rijke tube with a distributed heat source have been investigated. The heat release model consists of a row of distributed heat sources with individual heat release rates. The integrated heat release rate is then coupled with the acoustic perturbation for thermoacoustic analysis. Unlike the conventional approach utilizing the Galerkin method to simulate the acoustic field in time domain, a dynamic system is developed by spatially discretizing the acoustic equations using standard finite difference schemes in a Method of Lines (MOL) manner. In addition, a continuation approach is employed to analyze the nonlinear characteristics inherent to the heat-acoustic interaction. Hopf bifurcations and limit cycles are captured to delineate the nonlinear stability of the system. This methodology is first validated and shown to yield good predictions. Influences of multiple heat sources, time delay and heat release distribution are then studied to reveal the extensive nonlinear characteristics involved in the case of a distributed heat source. It is found that distributed heat source plays an important role in determining the stability of a thermoacoustic system.

Introduction

Thermoacoustic instability has been a serious impediment to develop NOx tolerant combustion systems both for aircraft propulsion and power generation gas turbines including rocket motors, industrial burners etc[1-3]. It arises from the interaction between the heat release and acoustic pressure or velocity oscillations within the combustion system. Rijke tube, a typical time-delayed thermoacoustic system, is a classical tool employed for the study of thermoacoustic instability. It usually consists of an open-end tube and heat source inside it. When the heat source is placed in certain positions along the tube, sound would emit from the tube. The sound is generated due to the transfer from unsteady heat release to acoustic energy. Despite the simplicity in structure, it contains rich nonlinear behaviors, which make it an excellent example for the study of thermoacoustic instability [4-6].

Most previous research, has focused on the classical Rijke tube with a single compact heat source, either a flame or a hot-wire gauze for the fact that the heat source is small enough compared with the acoustic wavelength. Heckl [7] developed empirical models for the nonlinear behavior of both heat release and the reflection coefficients based on experimental results. Limit cycle amplitudes were predicted and flow reversal at the heat source and nonlinear effects at the tube ends were demonstrated to be important in limiting the amplitude. Matveev [8-9] combined linear theory and thermal analysis to predict the linear stability boundaries in a horizontal Rijke tube. A special form of the nonlinear heat transfer function was introduced to extend the method to nonlinear stability analysis. Hysteresis phenomenon was reported in the stability boundary and limit cycles were predicted as observed in experiments. Heckl and Howe [10] conducted stability analysis of the Rijke tube by making use of a Green's function. Oscillations were described in terms of the eigenmodes of an integral equation derived using the Green's function and the predictions of stability behavior were in line with Rayleigh's criterion. Balasubramanian and Sujith [11] studied the role of non-normality and nonlinearity in thermoacoustic system in a Rijke tube using the heat release model from Heckl [7]. It was shown that the non-normality inherent in the thermoacoutic system could result in transient growth of oscillations which can trigger nonlinearities in the system. Noble *et al*. [12] described a data-driven nonlinear and chaos theory–based analysis of thermoacoustic instabilities in a simple Rijke tube. It only relied on experimental data with no implicit assumptions. PLIFH measurement of OH radical at the rate of 2500 Hz was used to capture the thermoacoustic instability modes appeared in the Rijke tube. Chaotic behavior was identified in the thermoacoustic instability. Juniper [13] employed adjoint looping of the nonlinear governing equations as well as an optimization routine to find the lowest initial energy of a Rijke tube. This state could trigger self-sustained oscillations and was known as the 'most dangerous' initial state. It was found that self-sustained oscillations can be reached over approximately half the linearly stable domain.

However, in practical combustion systems, the flame scale is usually considerable under most circumstances, especially with larger fuel flow or output power load, smaller excess air ratio, smaller nozzle spray angle etc. In this case, it is not quite appropriate to deal with the heat source as a single point source. Moreover, Heckl [14] reported that the heat source distribution has a first order influence on the stability of

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the thermoacoustic system. To the authors' knowledge, there is no prior research specifically focused on the bifurcation and nonlinear analysis of the Rijke tube with a distributed heat source. Thus, it is of primary concern in this paper.

Physical Model

In practice, the Rijke tube is often oriented vertically, in which a base flow is driven by natural convection. For the purpose of neglecting this complicated convection, in this paper, a primary concern is a horizontal Rijke tube to study the instability of thermoacoustic system.

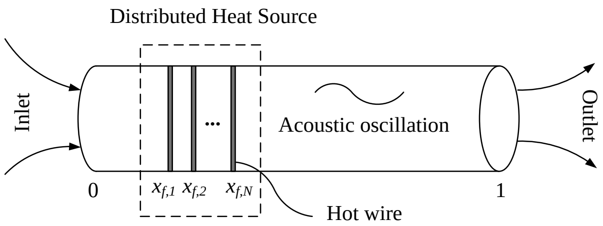


Fig.1 Schematic showing a horizontal Rijke tube with a distributed heat source (surrounded by dash line).

Figure 1 shows a schematic of a horizontal Rijke tube with a distributed heat source. The base flow driven by an external fan, passes through the tube and is heated up by hot wire gauzes (as discrete heat sources) placed along the tube at locations $x_{f,n}$, where n =1, 2, 3, …, N. Naturally, the tube displays an infinite number of acoustic modes. The thermal energy could be transferred to acoustic energy as long as they are in phase [15]. The fluid is treated as a perfect, inviscid and non-heat-conducting gas. Hence, the influence of the mean flow and mean temperature gradients can be excluded and the acoustic damping (ζ) principally arises from the acoustic boundary layer and acoustic radiation [8].

The heat release model consists of a row of distributed heat sources with individual heat release rates. The integrated heat release rate is then coupled with the acoustic perturbation for thermoacoustic analysis. Here, a modified form of King's law [7] is utilized as

$$Q_n = \beta\left(\left|\frac{u_0}{3}+u_{f,n}(t-\tau_n)\right|^{\frac{1}{2}}-\left(\frac{u_0}{3}\right)^{\frac{1}{2}}\right)\delta_D(x-x_{f,n}) \quad (1)$$

where β is the heat release coefficient representing the strength of the heat sources, incorporating all the details of the fluid, the hot wire gauze and the tube; u_0 is the velocity of the unperturbed base flow; τ_n is the time delay; $\delta_D(x-x_{f,n})$ is the Dirac Delta function to narrow the heat-release region specifically at the wire position $x_{f,n}$.

The integrated unsteady heat release is thus given as

$$Q=\sum_{n=1}^{N}Q_n=\sum_{n=1}^{N}\beta_n\delta_D(x-x_{f,n})\left(\left|\frac{1}{3}+u_{f,n}(t-\tau_n)\right|^{\frac{1}{2}}-\left|\frac{1}{3}\right|^{\frac{1}{2}}\right) \quad (2)$$

The generic acoustic damping coefficient [8] is defined as

$$\xi=\pi\left(\frac{S}{L_0^2}+\frac{L}{R_0}\sqrt{\frac{2}{\pi}}\frac{\sqrt{\nu}+\sqrt{\chi}(\gamma-1)}{\sqrt{c_0L_0}}\right) \quad (3)$$

where ν is the kinematic viscosity and χ is the thermal diffusivity.

Thus, the non-dimensional governing equations are

$$\frac{\partial u}{\partial t}+\frac{\partial p}{\partial x}=0 \quad (4)$$

$$\frac{\partial p}{\partial t}+\frac{\partial u}{\partial x}+\xi p-Q=0 \quad (5)$$

The boundary conditions are treated as $\partial u/\partial x = 0$ and $p = 0$ at $x = 0, 1$, which enable the problem of interest to be a well-posed initial value problem within time domain. The acoustic energy per unit volume is defined as $E= (u^2 + p^2)/2$ as done in Ref. 13 to demonstrate the acoustic amplitude inside the tube.

Numerical Scheme

To study dynamic behavior of the time-delayed system of Rijke tubes, a dynamic system constituting several ordinary differential equations (ODEs) should be constructed from the partial differential equations Eqns. (4) and (5). In the past, the Galerkin method [16] was extensively employed, for which two sets of presumed basis functions for u and p were used to convert Eqns. (4) and (5) to ODEs for each acoustic mode. However, in this paper, the approach originally employed is the Method of Lines (MOL) [17] to build up a dynamic system for the Rijke tube. The basic idea of the MOL is to directly discretize the spatial variable x in PDEs, but keep the temporal variable t be continuous. Thus, one could obtain a series of ODEs naturally without introducing any approximations. It should be pointed out that the resulting complexity of discretization and expensive computation cost are still affordable considering the availability of high computational resources nowadays.

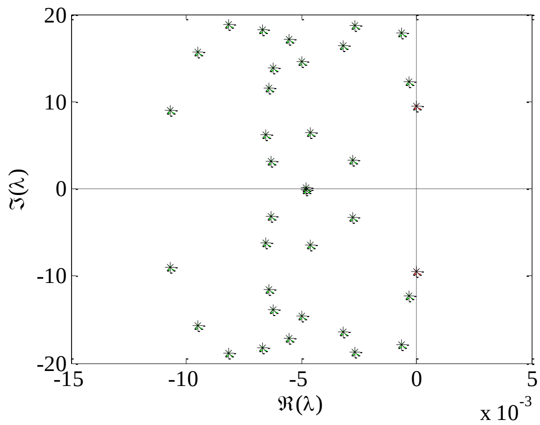
To obtain the unsteady solution structure, the governing equations Eqns. (4) and (5) were discretized by a finite difference method. The x domain along the length of the tube is divided into N points indexed as x_i (i = 1, 2, … , N). Thus there are N-1 parts with an identical interval of 1/(N-1). Second-order central difference scheme was used to discretize the spatial derivatives $\partial u/\partial x$ and $\partial p/\partial x$. Particularly, a second-order upwind difference scheme was employed to deal with the Neumann boundary conditions for u at x_1 and x_N, whereas specific values were given for the Dirichlet boundary conditions of p.

Thus, we could obtain the dynamic system written symbolically as

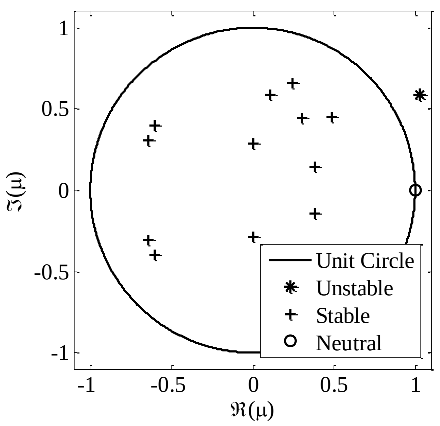
$$\frac{d\boldsymbol{y}}{dt}=\boldsymbol{f}(\boldsymbol{y},\tau,\boldsymbol{\theta}) \quad (6)$$

where $\boldsymbol{y}$ is the dependant variable vector of size $2N$, $\boldsymbol{f}$ is the vector function of size $2N$ and $\boldsymbol{\theta}$ is the bifurcation parameter vector ($= |\beta, \xi, x_f|$).

A numerical continuation method [18-19] was employed to investigate the bifurcation analysis of the time-delayed thermoacoustic system. First, the system was linearized around the pre-determined steady state (equilibrium) solution and the corresponding eigenvalues of the linearized system, namely the roots of the characteristic equation were calculated. These roots were first approximated by a linear multistep method (LMS-method) and then corrected using a Newton iteration method [19]. Mathematically, the rightmost characteristic root, or the characteristic root with the maximal real part, conclusively indicates whether or not the system is stable. If the rightmost characteristic root crosses zero as a bifurcation parameter varies, bifurcation may arise, as shown in Fig.2 (a). Furthermore, a prediction-correction approach [20-21] was employed to capture all the branches starting from the bifurcation point. Moreover, the Floquet-multiplier scheme was utilized to determine the stability of each branch, for which the solution is stable only if the moduli of all the multipliers are less than unity, as demonstrated in Fig. 2(b).



(a)



(b)

Fig.2 Numerical method: (a) Characteristic roots at a Hopf point; (b) Floquet Multipliers for an unstable periodic solution.

Results and Discussions

In this section, influences of multiple heat sources and time delays are studied to reveal the extensive nonlinear characteristics involved in the case of a distributed heat source.

Firstly, this numerical approach was validated through comparison with the reported experimental data [8, 22] and numerical calculations [23] in the case of a single heat source. From Figure 3, it can be concluded that the current approach yields very good prediction of the stability boundary and is more accurate than the results provided in Ref.23 using the Galerkin method.

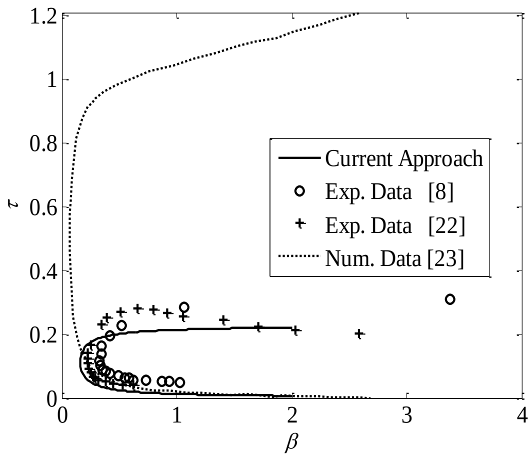


Fig.3 Comparison of numerical calculations with experimental data for a single heat source. Here ζ = 0.0281 and $x_{f,1}$ = 0.25.

The effect of varying the heat release parameter β on the dynamic behavior of the system was presented by the bifurcation diagram shown in Fig. 4.

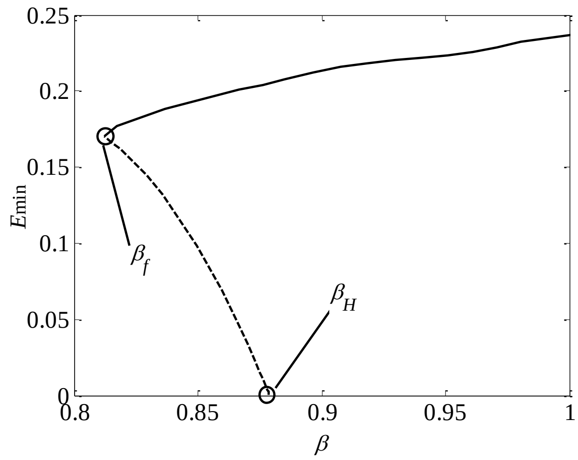


Fig.4 Bifurcation diagram of minimum acoustic energy $E_{\min}$ versus β for a single heat source. Here ζ = 0.043, $x_{f,1}$ = 0.36 and τ = 0.02. The solid line represents the stable solutions and the dash line is for the unstable ones.

Figure 4 shows a subcritical Hopf bifurcation and is similar to the bifurcation diagrams reported in [13, 23]. The stability of each periodic solution was determined by Floquet multipliers, as discussed in above section.

The steady state solution is stable for $\beta < 0.879$. At the Hopf bifurcation point β_H ($\beta_H = 0.879$), the system loses stability and small amplitude periodic solutions (limit cycle) emerge from it. These limit cycles are unstable and will be stabilized through a 'turning point' or referred to as a fold bifurcation point ($\beta_f = 0.812$). When $\beta < \beta_f$, the steady state solution is stable for perturbations of any magnitude. In the region of $\beta_f < \beta < \beta_H$, linearly stable steady state solution, small-amplitude unstable periodic solutions and large-amplitude stable limit cycles coexist to form a bistable region.

As is well known, the generation of sound in a Rijke tube is a result of the standing acoustic wave being set up in the tube [24]. The heat source plays the role of exciting the duct mode acoustic waves as well as reinforcing and sustaining the excited waves. According to the Rayleigh criterion [15], "when the pressure oscillation is in phase with the unsteady heat release, thermal energy could be transferred to acoustic energy and the acoustic oscillations can be further strengthened".

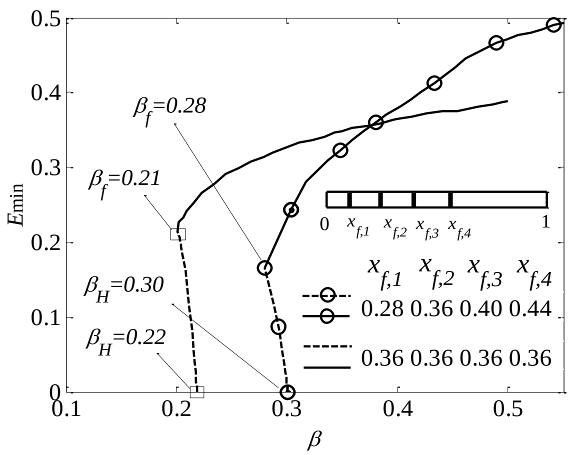


Fig.5 Bifurcation diagrams of minimum acoustic energy $E_{\min}$ as a function of β with four different heat sources concentrated and distributed at different positions along the tube. Here $\zeta = 0.043$ and $\tau = 0.02$. The solid line represents the stable solutions and the dash line is for the unstable ones.

The influence of multiple heat sources was investigated as shown in Figure 5. It can be seen that there are striking differences regarding β_f and β_H between distributed heat sources and concentrated heat sources (refer to the different positions along the tube). In the presence of distributed heat sources, β_f and β_H have increased by 33.3% and 36.4% respectively, which means the system has become more stable. This is due to that the acoustic perturbation excited by the foregoing heat source could interact with the subsequent unsteady heat release in the case of distributed heat sources. As long as they are (partly) in phase, the heat would be added during compression and removed during expansion. As a result, the acoustic perturbation can be further strengthened. Otherwise, it would be damped. In other words, as the integrated heat release acts as a source of acoustic energy, the thermoacoustic system becomes less stable, vice versa.

The influence of time delay was explored as shown in Figure 6. Subcritical Hopf bifurcations were also observed with different time delays. As the time delay remains the same at one heat source, the increase at the other heat source would destabilize the system (see τ = 0.02, 0.02 and τ = 0.02, 0.04), whereas the decrease would stabilize the system significantly (see τ = 0.02, 0.02 and τ = 0.01, 0.02). Moreover, the proportional increase or decrease of both time delays would make the system less or more stable respectively (see τ = 0.01, 0.02 and τ = 0.015, 0.03). Since a time delay is the time taken for the change of acoustic velocity to get reflected in heat release perturbation [24], a decreased time delay would lead to a faster heat transfer between the heat source and air flow. Consequently, the air would quickly attain the same temperature as the heat source, which eventually reduces and eliminates the heat transfer and diminishes the acoustic oscillation.

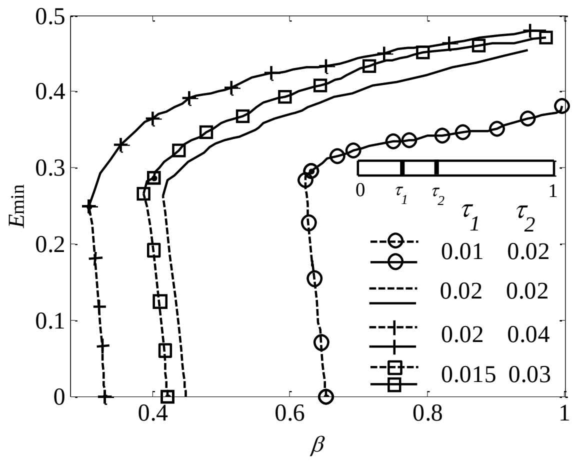


Fig.6 Bifurcation diagrams of minimum acoustic energy $E_{\min}$ versus β. Here $\zeta = 0.043$, $x_{f,1} = 0.26$ and $x_{f,2} = 0.36$. The solid line represents the stable solutions and the dash line is for the unstable ones.

Conclusions

In this study, the dynamic behaviors of a horizontal Rijke tube with a distribute heat source are studied in detail. Method of Lines (MOL) technique is employed to discretize the governing equations and a numerical continuation method is used for the study of thermoacoustic instability within the context of bifurcation and nonlinear analysis. It is found that distributed heat source plays an important role in determining the stability of a thermoacoustic system. The results reveal that the distribution of multiple heat sources plays an important role in the stability of system. It could either strengthen or weaken the acoustic energy based on specified distribution. Besides, the decreased (increased) time delay is proved adequately to have a stabilizing (destabilizing) effect on the system.

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